Parabolics and parahorics

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This is an expository paper on the internal structure of parabolic and parahoric subgroups. Most of the results are well-known, but I have never seen them treated systematically in one place in the literature. The one new result is due to Reeder and Yu, on the subquotients on the Moy-Prasad filtration of a parahoric subgroup over a local field. We summarize it here, in the simplest case.

Let A be a complete discrete valuation ring with quotient field k. Let π be a uniformizing parameter in A, and assume that the residue field $\mathfrak{f} = A/\pi A$ is finite. Let $\mathbf{G}$ be a split reductive group of rank ℓ over A, which is absolutely simple and simply-connected. The additive group of k has a locally compact topology coming from the valuation, and the group $\mathbf{G}(k)$ inherits a locally compact topology for which the subgroup $\mathbf{G}(A)$ is compact. In fact, $\mathbf{G}(A)$ is a maximal compact subgroup of $\mathbf{G}(k)$.

Unlike the case of real Lie groups, in this case there are ℓ further conjugacy classes of maximal compact subgroups of $\mathbf{G}(k)$. They are the maximal parahoric subgroups, which stabilize vertices in the building of $\mathbf{G}$ over k. We will construct representatives of these conjugacy classes containing a fixed Iwahori subgroup, using the affine root system of $\mathbf{G}$, as well as representatives of the $2^{\ell+1}-1$ conjugacy classes of all parahoric subgroups of $\mathbf{G}(k)$ which arise from their intersections. Each parahoric subgroup P can be identified with the A-valued points of a smooth group scheme $\mathbf{G}_P$ over A with general fiber $\mathbf{G}$ over k. Hence P has a descending filtration $P \triangleright P_1 \triangleright P_2 \triangleright \cdots$ where P_m is the subgroup which is the kernel of the reduction map $\mathbf{G}_P(A) \to \mathbf{G}_P(A/\pi^m A)$.

Moy and Prasad define a refinement of this filtration

$$P \triangleright P_{1/d} \triangleright P_{2/d} \triangleright \cdots$$

which depends on an integer $d \geq 1$ associated to P. Specifically, d is the denominator of the coordinates of the barycenter of the facet fixed by P in the building of $\mathbf{G}$, and $d = 1$ for $P = \mathbf{G}(A)$. The quotient $P/P_{1/d} = \mathbf{L}(\mathfrak{f})$ is the group of points of a split reductive group $\mathbf{L}$ of rank ℓ over $\mathfrak{f}$, and for $a \geq 1$ the quotients $P_{a/d}/P_{(a+1)/d} = V_a(\mathfrak{f})$ are the group of points of unipotent vector groups V_a over $\mathfrak{f}$. These afford linear representations of $\mathbf{L}$ over $\mathfrak{f}$ which are periodic with period d: $V_a \cong V_{a+d}$.

By the internal structure of P we mean the structure of the reductive quotient $\mathbf{L}$ and the representations V_a of $\mathbf{L}$ which arise from the subquotients of the Moy-Prasad filtration. Reeder and Yu have elucidated this structure using Vinberg's theory of torsion automorphisms. As a consequence, they show that the geometric quotient $V_a/\mathbf{L}$ is isomorphic to affine space of dimension $r(a)$ over $\mathfrak{f}$, and determine all cases when $r(a) \geq 1$. They also determine the parahorics P where V_1 has stable vectors for the action of $\mathbf{L}$, in the sense of geometric invariant theory. The existence of stable orbits on V_1 allows them to construct families of supercuspidal representations of depth $1/d$ of $\mathbf{G}(k)$, which are compactly induced from characters of $P_{1/d}$ which are trivial on $P_{2/d}$.

We begin with a review of roots and affine roots. All of this material is treated beautifully in Chapter VI of Bourbaki [**?**]. We then review what is known about $\mathbf{Z}$ and $\mathbf{Z}/d\mathbf{Z}$ gradings of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. After this preparation, we define the filtrations of parabolic and parahoric subgroups and discuss their internal structure, when $\mathbf{G}$ is split over k. In the last few sections, we review the theory of Bruhat and Tits, and describe the internal structure of parahoric subgroups of a simple, simply-connected group $\mathbf{G}$ which is split by a tamely ramified extension of k.

1 Roots and affine roots

In this section, we will review the theory of root systems and affine root systems as presented in Chapter VI of Bourbaki, giving the relation with the simply-connected, absolutely simple, split group $\mathbf{G}$ over A.

Let $\mathbf{S}$ be a maximal split torus in $\mathbf{G}$, let $X = \mathrm{Hom}(\mathbf{G}_m, \mathbf{S})$ be the cocharacter group of S and let $Y = \mathrm{Hom}(\mathbf{S}, \mathbf{G}_m)$ be its character group. There is a non-degenerate pairing

$$\langle x, y\rangle : X \times Y \to \mathbf{Z} = \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m)$$

Let R be the set of roots of $\mathbf{G}$. This is the finite subset of non-zero elements α in Y which occur in the representation of $\mathbf{S}$ on the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. We have the Cartan decomposition of the Lie algebra, as a representation of the torus $\mathbf{S}$:

$$\mathfrak{g} = \mathfrak{s} \oplus \sum \mathfrak{g}_\alpha$$

where the Lie algebra $\mathfrak{s}$ of $\mathbf{S}$ gives ℓ copies of the trivial character, and each root space $\mathfrak{g}_\alpha$ is one dimensional.

Let $\mathbf{Z}(\mathbf{G}) \subset \mathbf{S}$ be the center of $\mathbf{G}$ and $\mathbf{S}^* = \mathbf{S}/\mathbf{Z}(\mathbf{G})$ be the corresponding maximal torus in the adjoint group $\mathbf{G}^* = \mathbf{G}/\mathbf{Z}(\mathbf{G})$. The character group Y^* of $\mathbf{S}^*$ is the $\mathbf{Z}$-submodule of Y spanned by the roots, and the cocharacter group X^* of $\mathbf{S}^*$ is the subgroup of η in $X \otimes \mathbf{Q}$ which take integral values on R. The quotient group Y/Y^* is the Cartier dual of the center $\mathbf{Z}(\mathbf{G})$, which is of multiplicative type, and we have a perfect duality

$$X^*/X \times Y/Y^* \to \mathbf{Q}/\mathbf{Z}$$

.

There is also a set of coroots $R^\vee$ inside of X, defined using the theory of SL_2, which is in bijection with the set of roots. Each root α gives a simple reflections s_α of X and a simple reflection $s_{\alpha^\vee}$ of Y, defined by

$$s_\alpha(x) = x - \langle\alpha, x\rangle\alpha^\vee \quad s_{\alpha^\vee}(y) = y - \langle\alpha^\vee, y\rangle\alpha$$

These reflections preserve the sets of co-roots and roots respectively. The Weyl group $W = W(R)$ is the subgroup of $\mathrm{Aut}(R)$ that they generate. The pairing $X \times Y \to \mathbf{Z}$ is W-invariant, and by inspection the group W acts trivially on the quotients X^*/X and Y/Y^*.

In chapter VI of Bourbaki, root systems are discussed independent of algebraic groups. Their notation is a bit different than ours. Here is the translation:

$$Q(R) = Y^*, P(R) = Y, Q(R^\vee) = X, P(R^\vee) = X^*.$$

Hence the quotient groups $P/Q = Y/Y^*$ and $P^\vee/Q^\vee = X^*/X$. Since our group $\mathbf{G}$ is assumed to be absolutely simple, R is an irreducible (reduced) root system of type $A_n\ n \geq 1, B_n\ n \geq 2, C_n\ n \geq 3, D_n\ n \geq 4, G_2, F_4, E_6, E_7, or E_8$.

Let $\mathbf{B}$ be a fixed Borel subgroup of $\mathbf{G}$ containing $\mathbf{S}$. Then $\mathbf{B}$ determines a subset of positive roots R^+. The positive roots are those which occur in the representation of $\mathbf{S}$ on the Lie algebra $\mathfrak{b}$ of $\mathbf{B}$. The choice of $\mathbf{B}$ also determines a root basis $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, and every positive root has a basis expansion $\alpha = \sum m_i(\alpha).\alpha_i$ with all coefficients $m_i \geq 0$. The Weyl group is a finite Coxeter group, generated by the simple reflections s_{α_i}.

The finite group $\mathrm{Aut}(R)/W(R) = \mathrm{Aut}(R, \Delta)$ is isomorphic to the symmetric group S_1, S_2, S_3, with the latter case occurring only when R is of type D_4. This group acts faithfully on the finite abelian quotient groups X^*/X and Y/Y^*, and the natural pairing $X^*/X \times Y/Y^* \to \mathbf{Q}/\mathbf{Z}$ is non-degenerate and $\mathrm{Aut}(R, \Delta)$-invariant. The full group $\mathrm{Aut}(R)$ is isomorphic to the semi-direct product $W.\mathrm{Aut}(R, \Delta)$, and the choice of a Borel gives the splitting.

The roots α give linear functionals on the real vector space $X \otimes \mathbf{R}$. A fundamental domain for the action of W on $X \otimes \mathbf{R}$ is given by the closed Weyl chamber where $\alpha_i(x) \geq 0$ for $i = 1, 2, \ldots, \ell$. We equip $X \otimes \mathbf{R}$ with the sup norm $|x| = Max\{|\alpha_i(x)|\}$ and let C be the compact spherical alcove which is the intersection of the closed Weyl chamber with the set $\{x : |x| = 1\}$. A facet F of C consists of the points x where a proper subset of the α_i vanish, and the barycenter x of F is the point where the remaining subset Σ of the basic roots satisfy $\alpha_i(x) = 1$. Since all roots take integral values on the barycenter, x is a cocharacter of the adjoint torus $\mathbf{S}^*$, and gives a homomorphism

$$x = \eta : \mathbf{G}_m \to \mathbf{S}^* \to \mathbf{G}^* \to \mathrm{Aut}(G).$$

For example, if x is the barycenter of the interior of the alcove C, then $\alpha_i(x) = 1$ for all simple roots, and $x = \eta$ is the co-character $\rho^\vee$ of $\mathbf{S}^*$ given by half the sum of the positive co-roots.

Let

$$\beta = \sum m_i \alpha_i$$

be the highest root. This is by definition the highest weight for $\mathbf{S}$ in the adjoint representation of $\mathbf{G}$. The multiplicities $m_i = m_i(\beta)$ in the basis expansion satisfy $m_i \geq 1$ for all i, and $\sum m_i = h - 1$, where h is the Coxeter number of $\mathbf{G}$. If α is any positive root, then the multiplicity $m_i(\alpha)$ of α_i in the basis expansion of α satisfies $0 \leq m_i(\alpha) \leq m_i(\beta)$.

We now define the affine root system associated to the group $\mathbf{G}$. Consider the set of affine linear functionals $\psi = \psi(\alpha, n) = \alpha + n$ on $X \otimes \mathbf{R}$, with α in R and n in $\mathbf{Z}$. We say that α is the gradient of the affine root $\psi(\alpha, n)$. The affine roots $\psi_i = \psi(\alpha_i, 0) = \alpha_i$ together with $\psi_0 = \psi(-\beta, 1) = 1 - \beta$ form a basis for the affine root system, and the reflections in the $\ell + 1$ affine hyperplanes where $\psi_i = 0$ generate an affine Coxeter group. Putting $m_0 = 1$, we have the relation

$$\sum m_i \psi_i = 1$$

The fundamental alcove C in $X \otimes \mathbf{R}$ is the compact region where the $\ell+1$ basic affine roots satisfy $\psi_i(x) \geq 0$. Since the basic roots $\alpha_i = \psi_i$ take non-negative values on C, the same is true for all of the positive roots. Since the highest root β satisfies $1 - \beta(x) = \psi_0(x) \geq 0$ on C, all positive roots take values $0 \leq \alpha(x) \leq 1$ on C.

A facet F of C consists of the points where a proper subset of the basic affine roots take the value 0. The barycenter x of the facet F is the point of F where the remaining basic affine roots in the complementary subset Σ of $\{0, 1, \ldots, \ell\}$ take the same (rational) value. This value is completely determined by the relation $\sum m_i \psi_i = 1$; it is equal to $1/d$ with $d = d(F)$ is the sum of the multiplicities m_i over the subset Σ.

For example, if the complement Σ consists of a single affine root ψ_j, then the facet is a vertex x of C and $\psi_j(x) = 1/m_j$. At the other extreme, if the facet F is the interior of C, so the complement Σ contains of all the basis affine roots, then the common value $\psi_i(x)$ at the barycenter is equal to $1/h$.

For a general barycenter x, the element $d.x$ of $X \otimes \mathbf{R}$ takes integral values on all of the roots, so is a cocharacter of the adjoint torus and gives a homomorphism

$$d.x = \eta : \mathbf{G}_m \to \mathbf{S}^* \to \mathbf{G}^* \to \mathrm{Aut}(G).$$

2 Z and $\mathbf{Z}/d\mathbf{Z}$-gradings

In this section, we consider the gradings induced a homomorphism

$$\eta : \mathbf{G}_m \to \mathrm{Aut}(\mathbf{G})$$

as well as by the restriction of η to the subgroup μ_d of d-torsion in $\mathbf{G}_m$.

The $\mathbf{G}_m$ action gives a $\mathbf{Z}$-grading of the Lie algebra

$$\mathfrak{g} = \sum_{a \in \mathbf{Z}} \mathfrak{g}(a),$$

where $\mathfrak{g}(a)$ is the subspace where t in $\mathbf{G}_m$ acts by multiplication by t^a. The reductive subgroup $\mathbf{G}(0)$ of $\mathbf{G}$ which is fixed by $\mathbf{G}_m$ contains $\mathbf{S}$ as a maximal split torus. It acts linearly on each subspace $\mathfrak{g}(a)$. For example, $\mathfrak{g}(a) = \mathfrak{g}(0)$ is the adjoint representation of $\mathbf{G}(0)$.

When a is non-zero, the subspace $\mathfrak{g}(a)$ decomposes as a representation of S as the direct sum of one dimensional root spaces:

$$\mathfrak{g}(a) = \sum_{\langle \eta, \alpha \rangle = a} \mathfrak{g}_\alpha.$$

When $a = 0$, and one must add the ℓ dimensional space $\mathfrak{s}$.

Proposition 2.1 *If $a \neq 0$, then every $\mathbf{G}(0)$ invariant polynomial on $\mathfrak{g}(a)$ is a constant.*

In fact, over an algebraically closed field, $\mathbf{G}_0$ has an open orbit on each summand $\mathfrak{g}(a)$ with $a \neq 0$.

Now consider the restriction of the homomorphism η to μ_d. This gives rise to a $\mathbf{Z}/d\mathbf{Z}$-grading of $\mathfrak{g}$

$$\mathfrak{g} = \sum_{a \in \mathbf{Z}/d\mathbf{Z}} \mathfrak{g}(a)$$

where$\mathfrak{g}(a)$ is the subspace of $\mathfrak{g}$ where ζ in μ_d acts by multiplication by ζ^a. The reductive subgroup $\mathbf{G}_0$ of $\mathbf{G}$ which is fixed by μ_d contains $\mathbf{S}$ as a maximal split torus and acts on each subspace $\mathfrak{g}(a)$. For example, $\mathfrak{g}(0)$ is the adjoint representation of $\mathbf{G}_0$.

When a is non-zero (modulo d), the subspace $\mathfrak{g}(a)$ decomposes as a representation of S as the direct sum of one dimensional root spaces:

$$\mathfrak{g}(a) = \sum_{\langle \eta, \alpha \rangle \equiv a} \mathfrak{g}_\alpha.$$

When $a = 0$, and one must add the ℓ dimensional space $\mathfrak{s}$.

These representations were studied by Vinberg when $\mathfrak{f} = \mathbf{C}$ and by Levy in the more general situation where the characteristic of $\mathfrak{f}$ is a good prime which does not divide d. Under these assumptions, they prove the following generalization of Chevalley's theorem (which is the case $d = 1$)

Proposition 2.2 *The $\mathbf{G}_0$-invariant polynomials on $\mathfrak{g}(a)$ form a polynomial ring, with $r(a)$ independent generators.*

As a corollary, the geometric quotient $\mathfrak{g}(a)/\mathbf{G}_0$ is affine space of dimension $r(a)$ over $\mathfrak{f}$. The dimensions $0 \leq r(a) \leq \ell$ are interesting invariants of the μ_d action.

3 The filtration of parabolics and parahorics

We first define the parabolic subgroups of $\mathbf{G}$, which are algebraic subgroups $\mathbf{P}$ over A with $\mathbf{G}/\mathbf{P}$ projective. More precisely, we define the $2^\ell - 1$ parabolic subgroups which contain the fixed Borel $\mathbf{B}$, as these represent the distinct conjugacy classes. They correspond bijectively to the barycenters x of facets F of the compact spherical alcove C. We recall that C is the intersection of the closed Weyl chamber defined by the inequalities $\alpha_i(x) \geq 0$ with the set $\{x : |x| = 1\}$, and a barycenter x satisfies $\alpha_i(x) = 1$ for a non-empty subset Σ of Δ and $\alpha_j(x) = 0$ the basic roots in $\Delta - \Sigma$ Hence all roots α take integral values on x and $x = \eta$ is a cocharacter of the adjoint torus. Hence η gives a homomorphism

$$x = \eta : \mathbf{G}_m \to \mathbf{S}^* \to \mathbf{G}^* \to \mathrm{Aut}(G)$$

and by results of the previous section, an integral grading $\mathfrak{g} = \sum_{a \in \mathbf{Z}} \mathfrak{g}(a)$.

For each root α, we let $\mathbf{U}_\alpha$ be the corresponding root group (isomorphic to $\mathbf{G}_a$) in $\mathbf{G}$. Then the parabolic $\mathbf{P} = \mathbf{P}_x$ is generated by the torus $\mathbf{S}$ and the root groups $\mathbf{U}_\alpha$, for those roots α which satisfy $\langle \eta, \alpha \rangle \geq 0$. We define a terminating descending filtration

$$\mathbf{P} \triangleright \mathbf{P}_1 \triangleright \mathbf{P}_2 \triangleright \cdots \mathbf{P}_m \triangleright 1$$

where for $a \geq 1$, $\mathbf{P}_a$ is the unipotent subgroup of $\mathbf{P}$ generated the root groups $\mathbf{U}_\alpha$ with $\langle \eta, \alpha \rangle \geq a$. The integer $m = \langle \eta, \beta \rangle$, where β is the highest root. The subgroup $\mathbf{P}_1$ is the unipotent radical of $\mathbf{P}$, and for $a \geq 1$ the quotient $V_a = \mathbf{P}_a/\mathbf{P}_{a+1}$ is a vector group, which is a direct sum of the root spaces $\mathfrak{g}_\alpha$ for those roots with $\langle \eta, \alpha \rangle = a$. Hence V_a is isomorphic to $\mathfrak{g}(a)$ as a representation of $\mathbf{S}$.

The group $\mathbf{P}/\mathbf{P}_1 = \mathbf{L}$ is the Levi quotient of $\mathbf{P}$, which is reductive of rank ℓ with maximal split torus $\mathbf{S}$. From the definition, it is easy to see that the elements of $\Delta - \Sigma$ give a root basis for $\mathbf{L}$. Hence $\mathbf{L}$ has the same root datum as the subgroup $\mathbf{G}_0$ coming from the grading, and the two reductive groups are isomorphic. One can then use the exponential map to prove the following

Proposition 3.1 *The representation V_a of $\mathbf{L}$ is isomorphic to the representation $\mathfrak{g}(a)$ of $\mathbf{G}_0$.*

It follows that $\mathbf{L}$ has no polynomial invariants on V_a, other than constants. The open orbit over a separably closed field shows that each V_a is a prehomogeneous vector space. Since the weights of V_a are all roots of $\mathbf{G}$, the representations which occur are very restricted. In particular, when $\mathbf{G}$ is simply laced, the abelianization V_1 of $\mathbf{P}_1$ is the direct sum of $\#\Sigma$ irreducible minuscule representations of $\mathbf{L}$, with lowest weights the roots in Σ and distinct central characters.

We now give the analogous definition and results for the parahoric subgroups which contain a fixed Iwahori (the elements of $\mathbf{G}(A)$ which reduce to the fixed Borel in $\mathbf{G}(\mathfrak{f})$). Recall the closed alcove C in $X \otimes \mathbf{R}$ which is defined by the inequalities $\psi_i(x) \geq 0$. Let x be the barycenter of a facet F of C. We define the parahoric subgroup $P = P_x$ fixing the facet F, as well as the Moy-Prasad filtration associated to the barycenter x, as follows. Fix a Chevalley structure on $\mathbf{G}$ over A, consisting of the split maximal torus $\mathbf{S}$ and an isomorphism $e_\alpha : \mathbf{G}_a \to \mathbf{U}_\alpha$ over A for every root α, where $\mathbf{U}_\alpha$ is the corresponding root group. Associated to each affine root $\psi(\alpha, n)$ we define the subgroup $U_\psi = e_\alpha(\pi^n A)$ of $\mathbf{U}_\alpha(k)$. Then P is the subgroup of $G(k)$ which is generated by $\mathbf{S}(A)$ and the subgroups U_ψ, for the affine roots which satisfy $\psi(x) \geq 0$.

For example, if x is the vertex of C where $\psi_0 = 1 - \beta$ takes the value 1, then $\alpha(x) = 0$ for all roots of $\mathbf{G}$, and $P = \mathbf{G}(A)$. At the other extreme, if x is the barycenter of the interior of C, then $0 < \alpha(x) < 1$ for all positive roots. Hence P is the subgroup of $\mathbf{G}(A)$ generated by $\mathbf{S}(A)$, $\mathbf{U}_\alpha(A)$ for positive roots, and $\mathbf{U}_\alpha(\pi A)$ for negative roots. This is just the Iwahori subgroup which reduces to the fixed Borel subgroup of $\mathbf{G}$ modulo π. Finally, if x is a barycenter where $\psi_0(x) = 1 - \beta(x) = 0$, then the parahoric P_x contains the unipotent subgroup $\mathbf{U}_{-\beta}(\pi^{-1}A)$. In particular, P is not contained in $\mathbf{G}(A)$.

To define the Moy-Prasad filtration $P_{a/d} = P_{x,a/d}$ with $d = d(F)$, we let S_n be the kernel of the reduction map $\mathbf{S}(A) \to \mathbf{S}(A/\pi^n A)$. Then $P_{a/d}$ is generated by S_n with $n \geq a/d$ and the subgroups U_ψ, for the affine roots which satisfy $\psi(x) \geq a/d$.

The quotient $\mathbf{L} = P/P_{1/d}$ is reductive and contains the maximal torus $\mathbf{S}$ over $A/\pi A = \mathfrak{f}$. A basis for its root system is given by the gradients of the basic affine roots ψ_i which vanish at x. When $a \geq 1$ is not divisible by d, the subquotient $V_a = P_{a/d}/P_{(a+1)/d}$ is the direct sum of one dimensional root spaces $\mathfrak{g}_\alpha$ over $\mathfrak{f}$ where $\alpha(x) + n = a/d$ for some integer n. This determines its structure as a representation of $\mathbf{S}$, and constrains its structure as a representation of $\mathbf{L}$. When a is divisible by d, so $a/d = n$ is an integer, one has to add the ℓ trivial root spaces coming from the Lie algebra of S over $\mathfrak{f}$.

Lemma 3.2 *The root spaces which occur in the representation of $\mathbf{S}$ on the subquotient V_a of $P = P_x$ are precisely those which occur in the component $\mathfrak{g}(a)$ of the $\mathbf{Z}/d\mathbf{Z}$-grading of $\mathfrak{g}$ over $\mathfrak{f}$, which comes* *from the restriction to μ_d of the co-character $\eta = d.x : \mathbf{G}_m \to \mathbf{S}^{ad}$. Hence these representations of* $\mathbf{S}$ *are isomorphic.*

Notice that $\eta = d.x$ is a co-character of $\mathbf{S}^{ad}$, as it takes the value 0 or 1 on each basic root α_i and hence takes integral values on all the roots. To identify the root spaces which occur in $\mathfrak{g}(a)$, note that whenever the identity $\alpha(x) + n = a/d$ holds for some integer n, then $\alpha(d.x) = \langle \eta, \alpha \rangle \equiv a$ modulo d. Hence the root spaces $\mathfrak{g}_\alpha$ which lie in V_a are precisely those which contribute to the eigenspace $\mathfrak{g}(a)$, and these $\mathbf{S}$ modules are isomorphic over $\mathfrak{f}$

The coincidence of these root spaces in the lemma led Reeder and Yu to a proof of the following stronger result, when the characteristic of k does not divide d.

Proposition 3.3 *Let x be the barycenter of the facet F of C fixed by P, so that the basic affine roots take the value 0 or $1/d$ on x. Let $\eta = d.x$ be the associated cocharacter of $\mathbf{S}^{ad}$. Then the subgroup $\mathbf{G}_0$ of $\mathbf{G}$ over $\mathfrak{f}$ which is fixed by $\eta(\mu_d)$ is isomorphic to the reductive quotient $\mathbf{L}$ of P. Moreover, for every $a \geq 1$ the subquotient $V_a = P_{a/d}/P_{(a+1)/d}$ is isomorphic as a representation of $\mathbf{L} \cong \mathbf{G}_0$ to the submodule $\mathfrak{g}(a)$ in the associated $\mathbf{Z}/d\mathbf{Z}$- grading of $\mathfrak{g}$ over $\mathfrak{f}$.*

This implies that each geometric quotient V_a/L is affine space, of dimension $r(a)$. Since the weights in the representations V_a of $\mathbf{L}$ are all roots of $\mathbf{G}$, one can show that the representations themselves are very restricted. For example, assume that $\mathbf{G}$ is simply laced and that $d > 1$. Then the representation V_1 is the direct sum of $\#\Sigma$ irreducible minuscule representations of $\mathbf{L}$, which are distinguished by their central characters. The lowest weights for these representations, as characters of $\mathbf{S}$, are the gradients of the simple affine roots in Σ.

4 A cohomological classification

In the previous section, we described the internal structure of the parahoric subgroups of a split, simply-connected, simple group $\mathbf{G}$ over a local field k with finite residue field, using Vinberg's theory of automorphisms of finite order. We now turn to the general case of a simply-connected, simple group $\mathbf{G}$ over$k$, assuming only that $\mathbf{G}$ splits over a tamely ramified extension of k, with ramification index e.

A remarkable fact, which was discovered by Bruhat and Tits, is that the geometry of the alcove C in the building of $\mathbf{G}$ over k, as well as the isomorphism classes of the reductive quotient $\mathbf{L}$ and the unipotent subquotients $V_a = P_{a/d}/P_{(a+1)/d}$ which occur in the Moy-Prasad filtration of a parabolic subgroup P, depend only on the residue field $\mathfrak{f}$ and two invariants of a combinatorial nature. The first is a simple affine diagram ${}^e\mathscr{R}$, which is a connected affine Coxeter graph of the type listed in Bourbaki [**?**] [Ch VI, Thm 4] together with an orientation chosen for each multiple edge. The second is a conjugacy class F in the finite group $\mathrm{Aut}({}^e\mathscr{R})$ of automorphisms of this affine diagram.. More precisely, the affine diagram ${}^e\mathscr{R}$ determines the internal structure of parahorics in $\mathbf{G}$ over the maximal unramified extension K of k, and the automorphism F of the affine diagram determines the descent of this structure to k.

We will show how one can compute the combinatorial invariants $({}^e\mathscr{R}, F)$ from the cohomological data describing the isomorphism class of $\mathbf{G}$ over k, and describe which invariants occur for the different isomorphism classes. Finally, we will show how to determine the internal parahoric structure of G from its combinatorial invariants.

We first review the classification of the simple, simply-connected groups $\mathbf{G}$ over k. The first invariant is the split group $\mathbf{G}_0$ over k which becomes isomorphic to $\mathbf{G}$ over the separable closure k^s. The simply-connected group $\mathbf{G}_0$ is determined up to isomorphism by its root system. Let $\mathbf{S}_0$ be a maximal split torus in $\mathbf{G}_0$, let X be the cocharacter group of $\mathbf{S}_0$ and Y be the character group. We recall that the set of roots R is the finite subset of Y of the non-trivial characters of $\mathbf{S}_0$ which occur in the adjoint representation. Since $\mathbf{G}_0$ is simple, the root system is reduced and irreducible of type $A_n\, n \geq 1, B_n\, n \geq 2, C_n\, n \geq 3, D_n\, n \geq 4, G_2, F_4, E_6, E_7, or E_8$.

Fix an isomorphism $f : \mathbf{G}_0 \to \mathbf{G}$ over the separable closure k^s. For an element σ in the Galois group of k^s over k, we have ${}^\sigma f = f \cdot a_\sigma$, where a_σ is a one cocycle with values in $\mathrm{Aut}(\mathbf{G}_0)(k^s)$. The class of a_σ in the pointed cohomology set $H^1(k, \mathrm{Aut}(\mathbf{G}_0))$ determines the isomorphism class of $\mathbf{G}$ over k. We break this cohomology class into two parts.

The first is the image q_σ of a_σ, as a cocycle with values in the quotient group $\mathrm{Out}(\mathbf{G}_0)(k^s)$. If we fix a pinning $(B_0, T_0, \{X_i\})$ of $\mathbf{G}_0$ over k, the finite group of pinned automorphisms over k maps isomorphically onto the group $\mathrm{Out}(\mathbf{G}_0)(k^s)$. Since the group of pinned automorphisms is isomorphic to the constant étale group $\mathrm{Aut}(R, \Delta)$ the class of q_σ gives a tamely ramified homomorphism

$$q : \mathrm{Gal}(k^s/k) \to \mathrm{Aut}(R, \Delta)$$

up to conjugation. This homomorphism, when lifted to a map to the pinned automorphisms, is a cocycle determining the quasi-split inner form $\mathbf{G}_q$ of $\mathbf{G}$ over k. Since $\mathrm{Aut}(\mathbf{G}_0)$ is the semi-direct product of the adjoint quotient $\mathbf{G}_0/\mathbf{Z}(\mathbf{G}_0)$ with the subgroup of pinned automorphisms, standard twisting arguments in non-commutative cohomology show that the fibre over q in the surjective map $H^1(k, \mathrm{Aut}(\mathbf{G}_0)) \to H^1(k, \mathrm{Out}(\mathbf{G}_0))$ can be identified with the $\mathrm{Out}(\mathbf{G}_q)(k)$ orbits on the set of classes c in $H^1(k, \mathbf{G}_q/\mathbf{Z}(\mathbf{G}_q))$, where $\mathbf{Z}(\mathbf{G}_q)$ is the center of the quasi-split group $\mathbf{G}_q$. The class c is the second cohomological invariant.

All of this is true over an arbitrary field k, but in our case Kneser's theorem shows that the coboundary induces an isomorphism of pointed sets

$$H^1(k, \mathbf{G}_q/\mathbf{Z}(\mathbf{G}_q)) \to H^2(k, \mathbf{Z}(\mathbf{G}_q)).$$

Moreover, we can compute the finite abelian group $H^2(k, \mathbf{Z}(\mathbf{G}_q))$ from the root system R of $\mathbf{G}$ using Tate duality. Let M_q be the Cartier dual of $\mathbf{Z}(\mathbf{G}_q)$, which is a finite, commutative, étale group scheme over k. Over the separable closure, $M_q(k^s) = P/Q = Y/Y^*$ is the quotient of the weight lattice by the root lattice. The Galois group acts via the homomorphism q with image J_q in $\mathrm{Aut}(R, \Delta)$. Tate's theorem then gives canonical isomorphisms

$$H^2(k, \mathbf{Z}(\mathbf{G}_q)) = \mathrm{Hom}(H^0(k, M_q), \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}((Y/Y^*)^{J_q}, \mathbf{Q}/\mathbf{Z}) = (X^*/X)_{J_q}$$

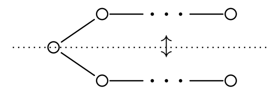
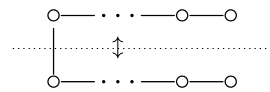
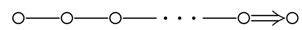
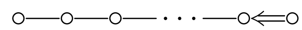
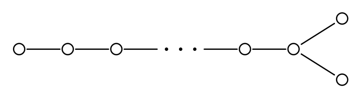
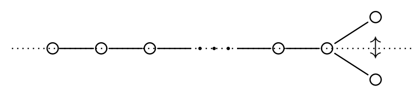
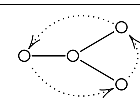
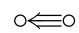
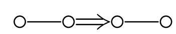
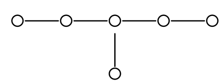
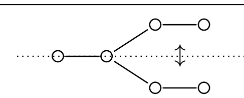
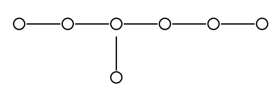
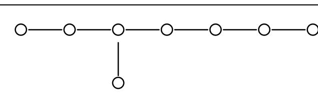
.

The group $\mathrm{Out}(\mathbf{G}_q)(k)$ is isomorphic to the centralizer of J_q in $\mathrm{Aut}(R, \Delta)$, and this centralizer acts naturally on the class c of $\mathbf{G}$ in $(X^*/X)_{J_q}$. Summarizing our results, we have proved the following.

Proposition 4.1 *The isomorphism class of the tamely ramified group* $\mathbf{G}$ *over the local field* k *is completely determined by the following data.*

- *The irreducible root system* R*,*
- *The tamely ramified homomorphism* $q : \mathrm{Gal}(k^s/k) \to \mathrm{Aut}(R, \Delta)$ *with image* J_q*,*
- *The orbit of the cohomology class* c *in the finite abelian group* $(X^*/X)_{J_q}$ *under the action of the centralizer of* J_q *in* $\mathrm{Aut}(R, \Delta)$.

Since the groups $\mathrm{Aut}(R, \Delta)$ and X^*/X are so small, we can tabulate all of the possibilities which occur. The image $J = J_q$ of the homomorphism q is a subgroup of S_1, S_2, S_3, so is determined by its order, which we tabulate as $\#J$.

R	$\#J$	$(X^*/X)_J$	Dynkin Diagram
A_{n-1}	1	$\mathbf{Z}/n\mathbf{Z}$	
A_{2n+1}	2	$\mathbf{Z}/2\mathbf{Z}$	
A_{2n}	2	1	
B_n	1	$\mathbf{Z}/2\mathbf{Z}$	
C_n	1	$\mathbf{Z}/2\mathbf{Z}$	
D_{2n}	1	$(\mathbf{Z}/2\mathbf{Z})^2$	
D_{2n+1}	1	$\mathbf{Z}/4\mathbf{Z}$	
D_n	2	$\mathbf{Z}/2\mathbf{Z}$	
D_4	$3, 6$	1	
G_2	1	1	
F_4	1	1	
E_6	1	$\mathbf{Z}/3\mathbf{Z}$	
E_6	2	1	
E_7	1	$\mathbf{Z}/2\mathbf{Z}$	
E_8	1	1	

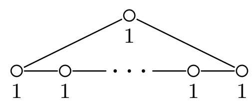
To illustrate, we will describe the tamely ramified groups $\mathbf{G}$ with root system $R = E_6$ over k. There is the split group $\mathbf{G}_0$ of rank 6, and for each tame quadratic field extension L of k there is the quasi-split group $\mathbf{G}_q$ of rank 4 which is split by L. Finally, when q is trivial, the group $\mathbf{G}_0$ has a inner form $\mathbf{G}$ of rank 2, corresponding to a non-trivial class c in $X^*/X = \mathbf{Z}/3\mathbf{Z}$. Both non-trivial classes in X^*/X lie in the same $\mathrm{Aut}(R, \Delta) = \mathbf{Z}/2\mathbf{Z}$ orbit. As a consequence,when $\mathbf{G}$ has rank 2, the map $\mathrm{Aut}(\mathbf{G})(k) \to \mathrm{Out}(\mathbf{G})(k) = \mathbf{Z}/2\mathbf{Z}$ is not surjective.

5 The affine diagram and its automorphism

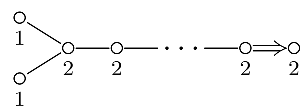
W recall that K be the maximal unramified extension of k, and let E be the unique cyclic, tamely ramified extension of K of degree e which splits the group $\mathbf{G}$. Since $\mathbf{G}$ is split over E, an alcove in the building of $\mathbf{G}$ over E is a simplex whose facets are given by non-empty subsets of the nodes of the extended Dynkin diagram $\mathscr{R}$ associated to the based root system (R, Δ). We can describe an alcove in the building of $\mathbf{G}$ over K using the techniques of tame descent. Since the cyclic group $\mathrm{Gal}(E/K)$ acts via an element of order e in the group $\mathrm{Aut}(R, \Delta)$ of pinned outer automorphisms of $\mathbf{G}$, the fixed affine root system has been computed by Reeder. The alcove of $\mathbf{G}$ over K is again a simplex, whose facets are given by non-empty subsets of the nodes of the affine diagram ${}^e\mathscr{R}$.

The affine diagrams which arise are listed below, with positive integers attached to the nodes. (These are the usual multiplicities of roots in the highest root when $e = 1$, and e times the multiplicities of roots in the highest short root, or twice the highest short root, when $e > 1$.) When $e = 1$ we write ${}^e\mathscr{R}$ simply as $\mathscr{R}$.

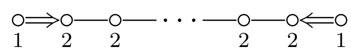
$\mathscr{A}_n$



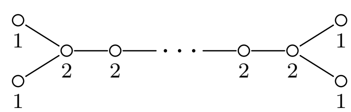
$\mathscr{B}_n$



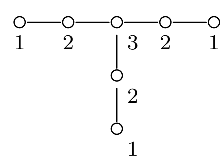
$\mathscr{C}_n$



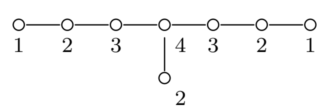
$\mathscr{D}_n$



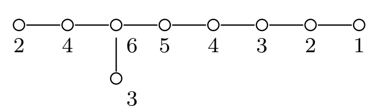
$\mathscr{E}_6$



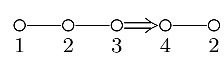
$\mathscr{E}_7$



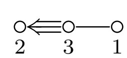
$\mathscr{E}_8$



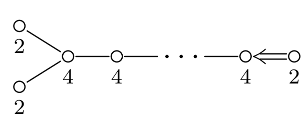
$\mathscr{F}_4$



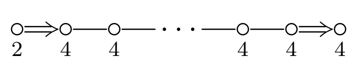
$\mathscr{G}_2$



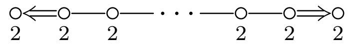
${}^2\mathscr{A}_{2n+1}$



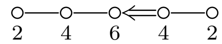
${}^2\mathscr{A}_{2n}$



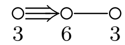
$^2\mathscr{D}_n$



$^2\mathscr{E}_6$



$^3\mathscr{D}_4$

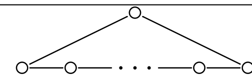
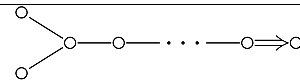
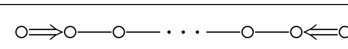
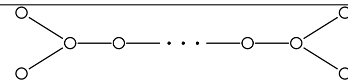
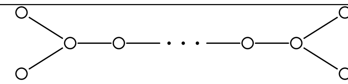
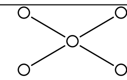
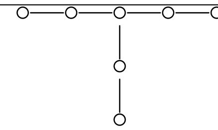
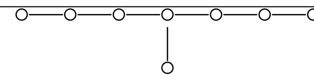
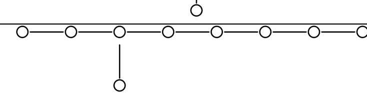
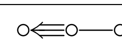
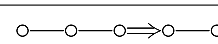
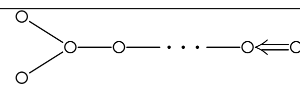
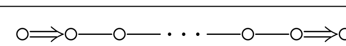
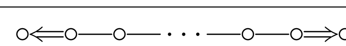
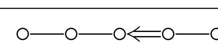
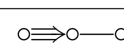


We obtain the alcove of $\mathbf{G}$ over k by unramified descent. The group $\mathbf{G}$ determines a homomorphism

$$\mathrm{Gal}(K/k) \to \mathrm{Aut}({}^e\mathscr{R})$$

and the image F of Frobenius is a well-defined conjugacy class in the finite group $\mathrm{Aut}({}^e\mathscr{R})$. The pair $({}^e\mathscr{R}, F)$ determines the geometry of the alcove C over k. We will see later how it determines the graded parahoric structure.

To enumerate the pairs $({}^e\mathscr{R}, F)$ which occur, we need to know the automorphism group of each affine diagram. When $e = 1$ the group $\mathrm{Aut}(\mathscr{R})$ is the semi-direct product $(X^*/X) \rtimes \mathrm{Aut}(R, \Delta)$, and when $e > 1$ the group $Aut({}^e\mathscr{R})$ is the abelian quotient $(X^*/X)_{\mathrm{Gal}(E/K)}$. These finite groups are tabulated below, where we use T_{2n} to denote the dihedral group of order $2n$ and S_n to denote the symmetric group on n letters. We give the semi-direct product decomposition when $e = 1$: the normal subgroup X^*/X acts simply-transitively on the nodes with multiplicity 1 and the canonical splitting is given by the subgroup fixing such a node.

${}^e\mathscr{R}$	$\mathrm{Aut}({}^e\mathscr{R})$	Affine Dynkin Diagram
$\mathscr{A}_{n-1}$	$T_{2n} = \mathbf{Z}/n\mathbf{Z} \rtimes S_2$	
$\mathscr{B}_n$	$\mathbf{Z}/2\mathbf{Z}$	
$\mathscr{C}_n$	$\mathbf{Z}/2\mathbf{Z}$	
$\mathscr{D}_{2n+1}$	$T_8 = (\mathbf{Z}/4\mathbf{Z}) \rtimes S_2$	
$\mathscr{D}_{2n}$	$T_8 = (\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_2$	
$\mathscr{D}_4$	$S_4 = (\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_3$	
$\mathscr{E}_6$	$S_3 = (\mathbf{Z}/3\mathbf{Z}) \rtimes S_2$	
$\mathscr{E}_7$	$\mathbf{Z}/2\mathbf{Z}$	
$\mathscr{E}_8$	1	
$\mathscr{G}_2$	1	
$\mathscr{F}_4$	1	
${}^2\mathscr{A}_{2n+1}$	$\mathbf{Z}/2\mathbf{Z}$	
${}^2\mathscr{A}_{2n}$	1	
${}^2\mathscr{D}_n$	$\mathbf{Z}/2\mathbf{Z}$	
${}^2\mathscr{E}_6$	1	
${}^3\mathscr{D}_4$	1	

6 From the cohomological invariants to the combinatorial data

To identify the alcove in the building of the $\mathbf{G}$, we need a recipe to pass from the cohomological invariants (R, q, c) to the combinatorial data $(\mathscr{R}, F)$.

The affine diagram is obtained from the root system R and the restriction of the homomorphism q to the inertia subgroup of $\mathrm{Gal}(k^s/k)$. If the image I_q of (tame) inertia in $\mathrm{Aut}(R, \Delta)$ is cyclic of order e, then the affine diagram has type ${}^e\mathscr{R}$. Note that non-isomorphic quasi-split groups $\mathbf{G}_q$ can have the same affine diagram, as the latter only depends on the order e of the tame inertia subgroup, whereas the isomorphism class of the group depends on the tamely ramified extension of k which splits $\mathbf{G}$.

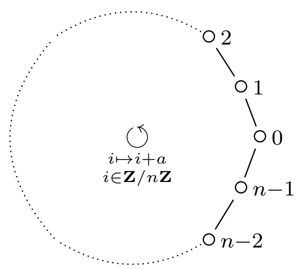
To determine the automorphism F of ${}^e\mathscr{R}$, we first assume that $e = 1$. Then the homomorphism q factors through $\mathrm{Gal}(K/k)$ and the image of Frobenius gives a conjugacy class s in the group $\mathrm{Aut}(R, \Delta)$.

If $s = 1$ the isomorphism class of $\mathbf{G}$ is determined by the $\mathrm{Aut}(R, \Delta)$ orbit of the cohomology class c in X^*/X. This gives a well-defined conjugacy class $F = c \times 1$ in the group $\mathrm{Aut}(\mathscr{R}) = (X^*/X) \rtimes \mathrm{Aut}(R, \Delta)$.

If $s \neq 1$ the group $\mathbf{G}$ is determined by the $\langle s \rangle$ orbit of the cohomology class c in the quotient group $(X^*/X)_{\langle s \rangle}$. If we lift c to a class c^* in X^*/X, then the product $F = c^* \times s$ gives a well-defined conjugacy class in $\mathrm{Aut}(\mathscr{R})$.

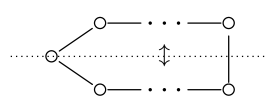
When $e > 1$, F is equal to the cohomology class c in $(X^*/X)_I = (X^*/X)_{J_q} = \mathrm{Aut}({}^e\mathscr{R})$.

With this recipe, we can easily identify the different isomorphism classes of groups associated to each pair $(\mathscr{R}, F)$. We describe them below, listing them in order of the affine diagrams.



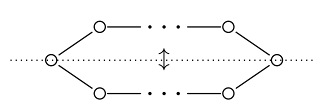
The affine diagram is $\mathscr{A}_{n-1}$. This is an n-gon, and F is a rotation by $\pm a$ units in T_{2n}. Write $a/n = b/m$ in lowest terms. There are m orbits on the nodes, and the rank of $\mathbf{G}$ over k is $m - 1$.

Let D be the division algebra of degree m^2 over k with invariant b/m. Then $\mathbf{G} = SL_{n/m}(D)$. This is the split form $\mathbf{G}_0 = \mathrm{SL}_n$ when $a \equiv 0$.



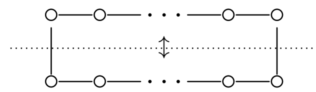
The affine diagram is $\mathscr{A}_{n-1}$ with $n = 2m + 1$. This is an n-gon and F is a reflection fixing a vertex in T_{2n}. There are $m + 1$ orbits on the nodes, and the rank of $\mathbf{G}$ over k is m.

Let L be the unramified quadratic extension of k and let W be a non-degenerate Hermitian space of odd rank $n = 2m + 1$ over L. Then $\mathbf{G} = SU(W)$. This is the unramified quasi-split form $\mathbf{G}_q$.



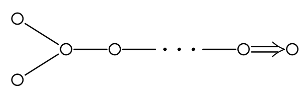
The affine diagram is $\mathscr{A}_{n-1}$ with $n = 2m$. This is an n-gon and F is a reflection fixing a vertex in T_{2n} (hence fixing the opposite vertex). There are $m + 1$ orbits on the nodes, and the rank of $\mathbf{G}$ over k is m.

Let L be the unramified quadratic extension of k and let W be a non-degenerate Hermitian space of even rank $n = 2m$ over L which contains an isotropic subspace of dimension m. Then $\mathbf{G} = SU(W)$. This is the unramified quasi-split form $\mathbf{G}_q$.



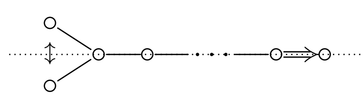
The affine diagram is $\mathscr{A}_{n-1}$ with $n = 2m$. This is an n-gon and F is a reflection through the midpoint of an edge in T_{2n}. There are m orbits on the nodes, and the rank of $\mathbf{G}$ over k is $m - 1$.

Let L be the unramified quadratic extension of k and let W be a non-degenerate Hermitian space of even rank $n = 2m$ over L which does not contain an isotropic subspace of dimension m. Then $\mathbf{G} = SU(W)$.



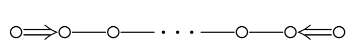
The affine diagram is $\mathscr{B}_n$ and F is trivial. There are $n + 1$ nodes and the rank of $\mathbf{G}$ over k is n.

Let W be a non-degenerate orthogonal space of odd dimension $2n + 1$ over k (the bilinear pairing is even and the associated quadratic form Q is non-singular) which contains an isotropic subspace of dimension n. Then $\mathbf{G} = \mathrm{Spin}(W)$. This is the split form $\mathbf{G}_0$.



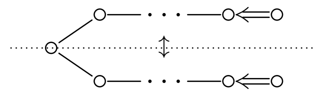
The affine diagram is $\mathscr{B}_n$ and F is the non-trivial involution. There are n orbits on the nodes and the rank of $\mathbf{G}$ over k is $n - 1$.

Let W be a non-degenerate orthogonal space of odd dimension $2n + 1$ over k (the bilinear pairing is even and the associated quadratic form Q is non-singular) which does not contain an isotropic subspace of dimension n. Then $\mathbf{G} = \mathrm{Spin}(W)$.



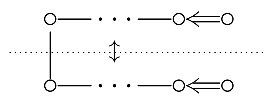
The affine diagram is $\mathscr{C}_n$ and F is trivial. There are $n+1$ nodes and the rank of $\mathbf{G}$ over k is n.

Let W be a non-degenerate symplectic space of even dimension $2n$ over k. Then $\mathbf{G} = Sp(W)$. This is the split form $\mathbf{G}_0$.



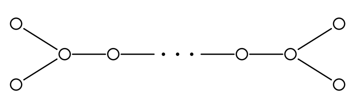
The affine diagram is $\mathscr{C}_n$ with $n = 2m$ and F is the non-trivial involution. There are $m+1$ orbits of the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be a non-degenerate Hermitian space over D of dimension n. Then $\mathbf{G} = U(W)$.



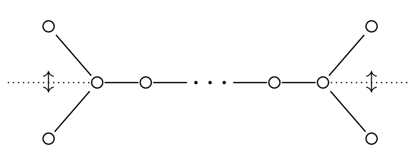
The affine diagram is $\mathscr{C}_n$ with $n = 2m+1$ and F is the non-trivial involution. There are $m+1$ orbits of the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be a non-degenerate Hermitian space over D of dimension n. Then $\mathbf{G} = U(W)$.



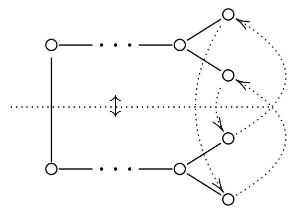
The affine diagram is $\mathscr{D}_n$ and F is trivial. There are $n+1$ nodes and the rank of $\mathbf{G}$ over k is n.

Let W be a non-degenerate orthogonal space of even dimension $2n$ which contains a maximal isotropic subspace of dimension n. Then $\mathbf{G} = \mathrm{Spin}(W)$ and the center of the Clifford algebra is the split étale quadratic extension $L = k + k$. This is the split form $\mathbf{G}_0$.



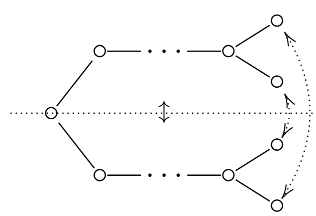
The affine diagram is $\mathscr{D}_n$ and F is the central element of order 2 in T_8. There are $n-1$ orbits on the nodes and the rank of $\mathbf{G}$ over k is $n-2$.

Let W be a non-degenerate even orthogonal space of even dimension $2n$ which does not contain an isotropic subspace of dimension $n-1$. Then $\mathbf{G} = \mathrm{Spin}(W)$ and the center of the Clifford algebra is the split étale quadratic extension $L = k + k$.



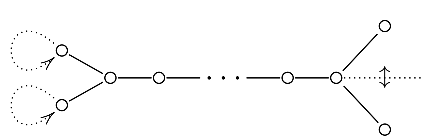
The affine diagram is $\mathscr{D}_n$ with $n = 2m+1$ and F is an element of order 4 in T_8. There are m orbits on the nodes and the rank of $\mathbf{G}$ over k is $m-1$.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra (in the sense of Tits [**?**] and Jacobsen[**?**]) is the split étale quadratic algebra $L = k + k$. Then $\mathbf{G} = U(W)$.



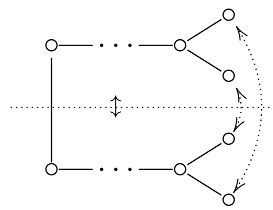
The affine diagram is $\mathscr{D}_n$ with $n = 2m$ and F is an element of order 2 in $P^\vee/Q^\vee$ which is not central in T_8. There are $m+1$ orbits on the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is the split étale quadratic algebra $L = k + k$. Then $\mathbf{G} = U(W)$.



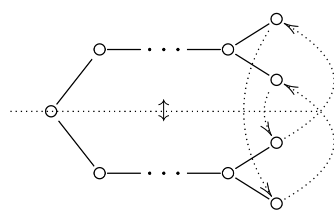
The affine diagram is $\mathscr{D}_n$ and F has order 2 in $\mathrm{Aut}(R, \Delta)$. There are n orbits on the nodes and the rank of $\mathbf{G}$ over k is $n-1$.

Let W be a non-degenerate orthogonal space of even dimension $2n$ where the center of the Clifford algebra is the unramified quadratic extension L of k. Then $\mathbf{G} = \mathrm{Spin}(W)$. This is the unramified quasi-split form $\mathbf{G}_q$.



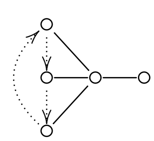
The affine diagram is $\mathscr{D}_n$ with $n = 2m+1$ and F is the remaining class of order 2 in T_8. There are $m+1$ orbits on the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is the unramified quadratic extension L of k. Then $\mathbf{G} = U(W)$.



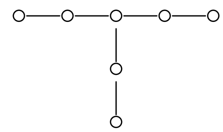
The affine diagram is $\mathscr{D}_n$ with $n = 2m$ and F has order 4 in T_8. There are m orbits on the nodes and the rank of $\mathbf{G}$ over k is $m-1$.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is the unramified quadratic extension L of k. Then $\mathbf{G} = U(W)$.



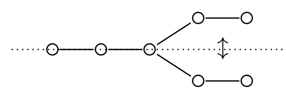
The affine diagram is $\mathscr{D}_4$ and F has order 3 in S_4. There are 3 orbits on the nodes and the rank is 2.

This is the unramified quasi-split inner form $\mathbf{G}_q$, split by a cubic extension.



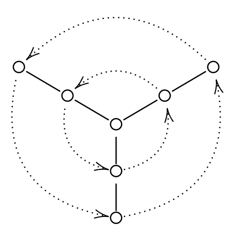
The affine diagram is $\mathscr{E}_6$ and F is trivial. There are 7 nodes and the rank is 6.

This is the split form $\mathbf{G}_0$.

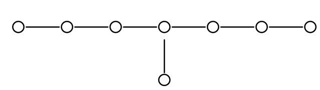


The affine diagram is $\mathscr{E}_6$ and F is a non-trivial involution in S_3. There are 5 orbits on the nodes and the rank is 4.

This is the unramified quasi-split form $\mathbf{G}_q$.

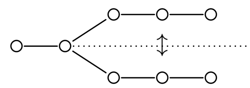


The affine diagram is $\mathscr{E}_6$ and F is an element of order 3 in S_3. There are 3 orbits on the nodes and the rank is 2.

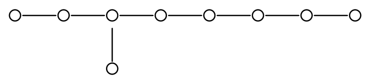


The affine diagram is $\mathscr{E}_7$ and F is trivial. There are 8 nodes and the rank is 7.

This is the split form $\mathbf{G}_0$.

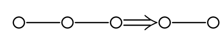


The affine diagram is $\mathscr{E}_7$ and F is the non-trivial involution. There are 5 orbits on the nodes and the rank is 4.



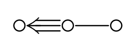
The affine diagram is $\mathscr{E}_8$ and F is trivial. There are 9 nodes and the rank is 8.

This is the split form $\mathbf{G}_0$.



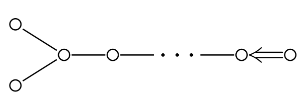
The affine diagram is $\mathscr{F}_4$ and F is trivial. There are 5 nodes and the rank is 4.

This is the split form $\mathbf{G}_0$.



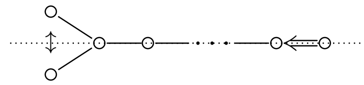
The affine diagram is $\mathscr{G}_2$ and F is trivial. There are 3 nodes and the rank is 2.

This is the split form $\mathbf{G}_0$.



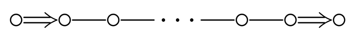
The affine diagram is ${}^2\mathscr{A}_{2m+1}$ and F is trivial. There are $m+1$ nodes and the rank of $\mathbf{G}$ over k is m.

Let L be a tamely ramified quadratic extension of k and let W be a non-degenerate Hermitian space of even rank $n = 2m$ over L which contains an isotropic subspace of dimension m. Then $\mathbf{G} = SU(W)$. This is a ramified quasi-split form $\mathbf{G}_q$.



The affine diagram is ${}^2\mathscr{A}_{2m+1}$ and F is the non-trivial involution. There are m orbits on the nodes and the rank of $\mathbf{G}$ over k is $m-1$.

Let L be a tamely ramified quadratic extension of k and let W be a non-degenerate Hermitian space of even rank $n=2m$ over L which does not contain an isotropic subspace of dimension m. Then $\mathbf{G}=SU(W)$.

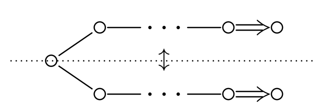


The affine diagram is ${}^2\mathscr{A}_{2m}$ and F is trivial. There are $m+1$ nodes and the rank of $\mathbf{G}$ over k is m.

Let L be a tamely ramified quadratic extension of k and let W be a non-degenerate Hermitian space of odd rank $n=2m+1$ over L. Then $\mathbf{G}=SU(W)$. This is a ramified quasi-split form $\mathbf{G}_q$.

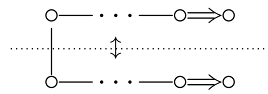
The affine diagram is ${}^2\mathscr{D}_n$ and F is trivial. There are n nodes and the rank of $\mathbf{G}$ over k is $n-1$.

Let W be a non-degenerate even orthogonal space of even dimension $2n$ where the center of the Clifford algebra is a tamely ramified quadratic extension L of k. Then $\mathbf{G}=\mathrm{Spin}(W)$. This is a ramified quasi-split form $\mathbf{G}_q$.



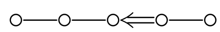
The affine diagram is ${}^2\mathscr{D}_n$ with $n=2m$ and F is the non-trivial involution. There are m orbits on the nodes and the rank of $\mathbf{G}$ over k is $m-1$

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is a tamely ramified quadratic extension L of k. Then $\mathbf{G}=U(W)$



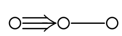
The affine diagram is ${}^2\mathscr{D}_n$ with $n = 2m + 1$ and F is the non-trivial involution. There are $m + 1$ orbits on the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is a tamely ramified quadratic extension L of k. Then $\mathbf{G} = U(W)$.



The affine diagram is ${}^2\mathscr{E}_6$ and F is trivial. There are 5 nodes and the rank is 4.

This is a tamely ramified quasi-split form $\mathbf{G}_q$.



The affine diagram is ${}^3\mathscr{D}_4$ and F is trivial. There are 3 nodes and the rank is 2.

This is a tamely ramified quasi-split form $\mathbf{G}_q$.

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7 From the combinatorial data to the graded parahoric structure

Having associated the combinatorial data $({}^e\mathscr{R}, F)$ to the group $\mathbf{G}$ over k, we want a recipe for determining its graded parahoric structure. Suppose that there are $\ell + 1$ orbits of the automorphism F on the nodes of the affine diagram ${}^e\mathscr{R}$. Then the group $\mathbf{G}$ has rank ℓ over k. The parahoric subgroups P of $\mathbf{G}(k)$ which contain a fixed Iwahori correspond bijectively to the non-empty F-stable subsets Σ of the nodes of ${}^e\mathscr{R}$. The integer $d = d(P)$ in the Moy-Prasad filtration

$$P \triangleright P_{1/d} \triangleright P_{2/d} \triangleright \cdots$$

is the sum, taken over the subset Σ, of the integral labels of the nodes.

The reductive quotient $\mathbf{L}$ of P also has rank ℓ over $\mathfrak{f}$. We can describe this quasi-split group over $\mathfrak{f}$ by giving its based root datum over the algebraic closure and the automorphism of the based root datum which determines the descent to $\mathfrak{f}$. First consider the case when $e = 1$. The simply-connected group $\mathbf{G}$ is split over K with based root system

$$\Delta \subset R \subset Q = \mathbf{Z}R \subset P = X^*(\mathbf{S})$$

where $\mathbf{S}$ is a maximal split torus over K. Let β be the highest root, and define the simple affine root $\psi_0 = 1 - \beta$. Then the nodes of the affine diagram $\mathscr{R}$ correspond to the simple affine roots ψ_0 as well as the affine roots $\psi_i = \alpha_i$ corresponding to the roots in Δ. If the parahoric P corresponds to the F-stable subset Σ of the affine roots, then the root basis of $\mathbf{L}$ over the algebraic closure of $\mathfrak{f}$ are the gradients of the complement of Σ in the set of simple affine roots. Since the subset Σ is stable under F, the automorphism F of the affine diagram gives an automorphism of this based root datum, which gives the descent of $\mathbf{L}$ to $\mathfrak{f}$.

The situation when $e > 1$ is similar. Here the nodes of the affine diagram ${}^e\mathscr{R}$ correspond bijectively to the orbits $[\psi_i]$ of the cyclic group $\mathrm{Gal}(E/K)$ on the root basis Δ, together with an additional orbit $[\psi_0]$ which occurs on the roots.

We can describe the representations $V_a = P_{a/d}/P_{(a+1)/d}$ of $\mathbf{L}$ which occur as subquotients of the Moy-Prasad filtration of P, in a similar manner. These depend only on the index a modulo d.